CATEGORY A

Problem 1

For the *10th* anniversary of ***IUHD***, scientists of ***IUHD*** invented "*OMOUScoin*" which is a new cryptocurrency and for testing there are 2024 wallets with $a_1, a_2, \ldots, a_{2024}$ positive integer amounts of *OMOUScoins* respectively. There is some hacker who found following weaknesses (operations) of the blockchain:

1. It is possible to steal an equal number of *OMOUScoins* from any pair of wallets.
2. It is possible to double amount of *OMOUScoins* in any wallet.

A. Analyze when it's possible to empty all wallets after a finite sequence of operations?

B. Operation 2 is now replaced with "Triple the amount of *OMOUScoins* in any wallet." Analyze when it's possible to empty all wallets after a finite sequence of operations?

Proposed by Pirmyrat Gurbanov, & Murat Chashemov, IUHD, Ashgabat, Turkmenistan

Solution

We analyze situation any pair m, n

A. Perform the following algorithm:

1 If $m = n$ remove all the balls from each bag.
2 Assume without loss of generality $m > n$. Remove $n - 1$ balls from each bag.
3 Double the bag with 1 ball in it to get 2 balls.
4 Remove 1 from each bag.
5 Repeat **3** and **4** until both bags have 1 ball.
6 Remove 1 ball from both bags.
It is clear that this algorithm terminates.

B. Let $m + n \equiv 1 (mod\ 2)$. Then $(m - k) + (n - k) \equiv m + n \equiv 1\ (mod\ 2)$ and $3m + n \equiv m + n \equiv 1 (mod\ 2)$. Thus, $m + n \neq 0$, so it is actually impossible.

Problem 1(Cat A,B) Marking Scheme

№	Steps	Points
1.	Finding algorithm and observing pairing………………………..……………………………...	5pts
2.	Finding invariants and for part B (Cat A) …………………………………………………...	3pts
3.	Empty cases of part A, B (Cat A)………………………………………………………….	2pts

Problem 2

Show that if A is a real $n \times n$ matrix such that $A + A^2A^T + (A^2)^T = 0$, then $A = 0$.

Proposed by Karen Keryan & Vazgen Mikayelyan, Yerevan State University, Armenia

Solution

On one hand from the given equation we obtain

$(A^3)^T = -(AA^T + A^2(A^2)^T) \leq 0.$

Hence $I - A^3$ is positive definite and invertible.

On the other hand we have $(A^2)^T = -A - A^2A^T$ and transposing we get $A^2 = -A^T - A(A^2)^T$. Hence we get

$$A^2 = -A^T + A(A + A^2A^T).$$

Thus $A^T = A^3A^T$, so $A^T(I - A^3) = 0$. This together with invertibility of $I - A^3$ implies that $A = 0$.

Problem 2(Cat A) Marking Scheme

№	Steps	Points
1.	Observing $A^T(I + A^3) = 0$ or $A^T(I - A^3) =$...	4pts
2.	Getting that $I + A^3$ or $I - A^3$ is positive defined ……..……………………………....	5pts

Problem3 Let an infinite sequence $D_n(n = 0, 1, 2, ...)$ of circle disks with radii r_n situate on a plane. It is known that

$$r_0 = \frac{2}{3} - \varepsilon_0, r_n = \frac{1}{n} - \varepsilon_n \ (n = 1,2, ...)$$

where $\varepsilon_0 \in \left(0, \frac{1}{12}\right)$, while $\varepsilon_n \in \left(0, \frac{1}{12n}\right)$.

1. Does there exist any sequence of disks which can be situated into a circle with diameter 3 without intersections? (provide your argument).
2. Is it always possible to place all disks into a square with side 3 without intersections? (prove either the impossibility or give an example of the location).

Proposed by Victor Voytitsky, RUDN University, Russian Federation

Solution

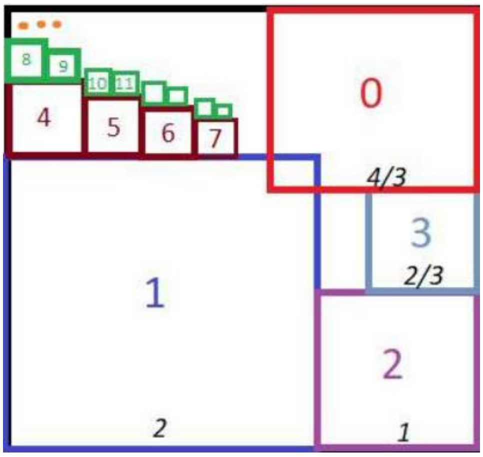
1. No, it doesn't exist. Really, if D_0 and D_1 are situated into a circle with radius 3 without intersections then distance between any two points on boundaries must be not greater than 3. On the other hand the boundary points on the straight line connecting the centers of the discs D_0 and D_1 are at the distance

$$d \geq 2r_0 + 2r_1 = \frac{4}{3} - 2\varepsilon_0 + 2 - 2\varepsilon_1 = 3 + \frac{1}{3} - (\varepsilon_0 + \varepsilon_1) > \frac{10}{3} - 2\left(\frac{1}{12} + \frac{1}{12}\right) = 3.$$

2. Yes, it is always possible. Each disc can be situated correspondingly inside squares $S_n (n = 0,1,2, \dots)$ with the sides $a_0 = \frac{4}{3}$; $a_n = \frac{2}{n} (n = 1,2, \dots)$. The largest S_1 and S_0 can be placed in the lower left and higher right corners of the square S with side 3 diagonally (see picture). This squares have a common square intersection area but even largest disks in squares can be situated without intersection. The remaining squares can be situated into S without intersections. Really, let squares S_2 and S_3 (with the sides 1 and $2/3$) be situated to the right side of S_1, and all the others are above S_1 by layers. The first layer is the four squares with sides $\frac{2}{4}, \frac{2}{5}, \frac{2}{6}, \frac{2}{7}$ (brown squares at the picture), the second layer is the eight squares with sides $\frac{2}{8}, \frac{2}{9}, \frac{2}{10}, \dots, \frac{2}{15}$ (green squares at the picture), and so on. Then the height of all layers will be $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots = 1$. From the obvious inequality $\frac{1}{2k} + \frac{1}{12k+1} < \frac{1}{k}$ we have

$$1 > \frac{1}{2} + \frac{1}{3} > \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} > \frac{1}{8} + \frac{1}{9} + \dots + \frac{1}{15} > \dots$$

So, all squares $S_4, S_5, \dots$ can be situated (without intersections) in the rectangular domain above the square S_1, and it is right for all discs inside them.

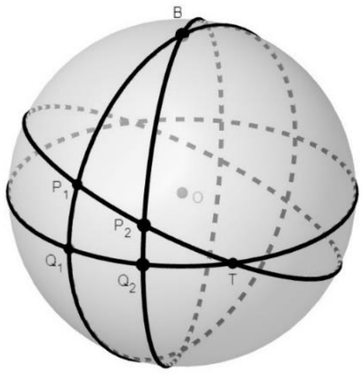


Problem 3(Cat A) Marking Scheme

№	Steps	Points
1.	Part a)…………………………………………………………………	2pts
2.	Part b) construction……………………………………………………...	8pts
3.	Reducing points for not complete construction………………………………..	-2pts

Problem 4

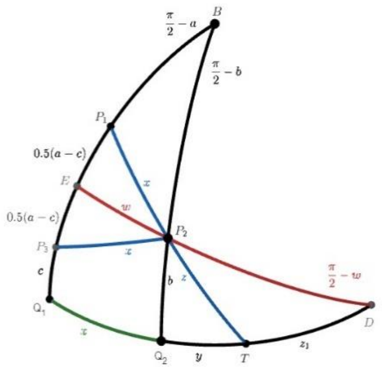
On a unit sphere with center O, a great circle and one of its poles B is drawn. Two great circles through B intersect the first great circle at Q_1 and Q_2 so that $Q_1Q_2 < \frac{\pi}{2}$. Another great circle intersects arcs BQ_1 and BQ_2 at points P_1 and P_2 so that $P_1P_2 = Q_1Q_2$. This great circle also intersects the first great circle at T (See Figure). Then $P_1T + TQ_2 = P_2T + TQ_1 = \frac{\pi}{2}$.



Proposed by Yagub Aliyev, ADA University, Baku, Azerbaijan

Solution:

Let us denote $P_1P_2 = Q_1Q_2 = x$. Since angles at Q_1 and Q_2 are right angles, by spherical sine theorem for triangle BQ_1Q_2, the angle between the two great circles at vertex B is also x (Whittlesey, page 30). Denote also $P_1Q_1 = a$, $P_2Q_2 = b$ (See Figure 5). Then $BP_1 = \frac{\pi}{2} - a$ and $BP_2 = \frac{\pi}{2} - b$. By spherical sine theorem for triangle BP_1P_2 (Whittlesey, page 115), $\frac{\sin B}{\sin P_1P_2} = \frac{\sin P_1}{\sin BP_2} = \frac{\sin P_2}{\sin BP_1}$.



Since $\sin B = \sin P_1P_2 = \sin x$,

$\sin\angle BP_1P_2 = \sin BP_2 = \cos b$, $\sin\angle BP_2P_1 \sin BP_1 = \cos a$. (*)

Take another point P_3 on the arc P_1Q_1 such that $P_2P_3 = x$. Such a point P_3 should exist, because $P_2Q_1 > x$. Denote the midpoint of P_1P_3 by E and let $Q_1P_3 = c$. Then $P_1E = EP_3 = 0.5(a-c)$. Since spherical triangle $P_1P_2P_3$ is isosceles, P_2E is its angle bisector and P_2E is perpendicular to P_1P_3. The three angles of the quadrilateral $EQ_1Q_2P_2$ are right angles. Therefore, the fourth $\angle EP_2Q_2$ is obtuse and $\angle EP_2B$ is acute. Since $\angle P_1P_2B < \angle EP_2B$, $\angle P_1P_2B$ is acute, too. Similarly, $\angle BP_3P_2 = \angle P_3P_1P_2$ is acute. Consequently, $\angle BP_1P_2$ is obtuse. So, $\angle BP_1P_2$ and $\angle BP_2P_3$ are obtuse, $\angle BP_2P_1$ and $\angle BP_3P_2$ are acute. Therefore by (*), $\angle BP_1P_2 = \frac{\pi}{2} + b$ and $\angle BP_2P_1 = \frac{\pi}{2} - a$. Similarly, $\angle BP_2P_3 = \frac{\pi}{2} + c$ and $\angle BP_3P_2 = \frac{\pi}{2} - b$. So, $\angle PP_2P_3 = \angle BP_2P_3 - \angle BP_2P_1 = c + a$. Therefore, $\angle P_1P_2E = \frac{c+a}{2}$.

Denote the intersection of the great circles EP_2 and Q_1Q_2 by D. Denote also $EP_2 = w, P_2T = z, Q_2T = y, TD = z_1$. Then $P_2D = \frac{\pi}{2} - w$ and $x + y + z_1 = \frac{\pi}{2}$. It remains only to show that $z = z_1$. This follows from the fact that spherical triangle P_2TD is isosceles. Indeed, $\angle D = Q_1E = \frac{a+c}{2}$ and $\angle TP_2D = \angle P_1P_2E = \frac{a+c}{2}$. Consequently, $\angle D = \angle TP_2D$ or $z = z_1$. Finally

$$P_1T + TQ_2 = P_2T + TQ_1 = x + y + z = \frac{\pi}{2}.$$

Problem 4 (Cat A) Marking Scheme

№	Steps	Points
1.	For part (*)	2pts
2.	$\angle P_1P_2E = \frac{c+a}{2}$.	5pts
3.	Last part with $P_1T + TQ_2 = P_2T + TQ_1 = x + y + z = \frac{\pi}{2}$	3pts

Problem 5

Let $n \geq 2$ be a natural number. Define the integral

$$I(n) := \int_0^{\pi/2} \frac{\{nx\}}{sinx} dx$$

where $\{s\}$ denotes the fractional part of the number s, i.e., $\{s\} \in [0, 1)$, and $s - \{s\}$ is an integer.

a) Prove that integral $I(n)$ is convergent.

b) Prove that there exist constants $C_1 > 0$ and $C_2 > 0$ such that for all $n \geq 2$ holds $C_1 \ln(n) \leq I(n) \leq C_2 \ln(n)$.

Proposed by Asen Bozhilov and Stoyan Apostolov, Sofia University St. Kliment Ohridski, Sofia, Bulgaria

Solution

a) The only point where the function is not defined (and hence not continuous) is the left endpoint 0. However, $\lim_{x\to\infty} \frac{\{nx\}}{sinx} = \lim_{x\to\infty} \frac{nx}{sinx} = n$ hence there is actually no singularity at 0.

Thus the integral is well-defined and convergent.

b) We have

$$I(n) = \int_0^{\frac{\pi}{2}} \frac{\{nx\}}{sinx} dx = \int_0^{\frac{\pi}{2n}} \frac{\{nx\}}{sinx} dx + \int_{\frac{\pi}{2n}}^{\frac{\pi}{2}} \frac{\{nx\}}{sinx} dx \leq n \int_0^{\frac{\pi}{2n}} \frac{x}{sinx} dx + \int_{\frac{\pi}{2n}}^{\frac{\pi}{2}} \frac{\pi}{2x} dx \leq n \frac{2\pi}{2n} + \frac{\pi}{2} lnx \Big|_{\pi/(2n)}^{\pi/2} = \pi + \frac{\pi}{2} lnn \leq 4 + 2lnn,$$

where for the first integral we used the inequalities $\{nx\} \leq nx$, $\frac{x}{\sin x} \leq 2$ for $x \in \left(0, \frac{\pi}{2}\right)$, and for the second one $\{nx\} \leq 1$ and $\frac{1}{\sin x} \leq \frac{\pi}{2x}$ for $x \in \left(0, \frac{\pi}{2}\right)$. Thus $I(n) \leq 10\, ln\, n$, since the inequality $4 + 2\, ln\, n \leq 10 ln\, n$ holds for all $n \geq 2$. So we may take $C_2 = 10$.

For the other inequality we have

$$I(n) = \int_0^{\pi/2} \frac{\{nx\}}{sinx} dx \geq \int_0^1 \frac{\{nx\}}{sinx} dx \geq \int_0^1 \frac{\{nx\}}{x} dx$$

where we used $\frac{1}{\sin x} \geq \frac{1}{x}$ for $x \in [0, 1]$. Now we have

$$\int_0^1 \frac{\{nx\}}{x} dx \sum_{k=0}^{n-1} \int_{k/n}^{(k+1)/n} \frac{\{nx\}}{x} dx = \sum_{k=0}^{n-1} \int_{k/n}^{(k+1)/n} \frac{nx - k}{x} dx = 1 + \sum_{k=1}^{n-1} (1 - k(\ln(k+1) - lnk))$$
$$= n - (n-1)\ln(n) + \ln((n-1)!)$$

Applying the Cesaro-Stolz theorem we obtain

$$\lim_{n\to\infty} \frac{1}{lnn} \int_0^1 \frac{\{nx\}}{x} dx = \lim_{n\to\infty} \frac{n - (n-1)lnn + \ln((n-1)!)}{lnn} = \lim_{n\to\infty} \frac{1 + nln\left(\frac{n}{n+1}\right)}{\ln(n+1) - \ln(n)} = \frac{1}{2}$$

This means that there exists n_0 such that $I(n) \geq \frac{1}{3} \ln n$ for all $n \geq n_0$. Since $I(n) > 0$ for all n, we have that $\mu := \min\{I(n) \mid n \leq n_0\} > 0$. Thus $I(n) > \frac{\mu}{ln(n_0)} ln\, n$ for all $n < n_0$. Thus we may take $C_1 = \min\left\{\frac{1}{3}, \frac{\mu}{\ln(n_0)}\right\}$.

Problem 5 (Cat A) Marking Scheme

№	Steps	Points
1.	Showing a) part ……………………………………………………………………......….	2pts
2.	$I(n) \leq 4 + 2lnn$ ……………………………………………………………..………......	1pts
3.	$I(n) \geq \int_0^1 \frac{\{nx\}}{x} dx$………………………………………………………..	1pts
3.	$\int_0^1 \frac{\{nx\}}{x} dx \sum_{k=0}^{n-1} \int_{k/n}^{(k+1)/n} \frac{\{nx\}}{x} dx = n - (n-1)\ln(n) + \ln((n-1)!)$ ………………	2pts
4.	$\lim_{n\to\infty} \frac{1}{lnn} \int_0^1 \frac{\{nx\}}{x} dx = \frac{1}{2}$ ………………………………………………	2pts
5.	Taking $C_1 = \min\left\{\frac{1}{3}, \frac{\mu}{\ln(n_0)}\right\}$…………………………………………………………	2pts

Problem 6 Let R be a ring, not necessarily having an identity element, such that

$$x + x^{2025} = x^{2024} + x^{10^5+11^2}$$

for each x in R. Prove that

1. R is a Boolean ring.
2. R is commutative.

Proposed by Gurtmyrat Motukov, ELTE, Budapest, Hungary

Solution

a) We first note that R has no non-trivial nilpotent elements: if $x^r = 0$ with $r \geq 1$ then $x = 0$. Assume by contradiction that $x \neq 0$ and $r \geq 2$ is the minimal positive exponent such that $x^r = 0$. By given equation

$$x^{r-1} = x^{10t+r-1} + x^{2t+r-2} - x^{2t+r-1} = 0$$

against the minimality of r. Let $I = x^{10t} - x^{2t} + x^{2t-1}$ then $Ix = x$. Moreover, it is easy to see by induction that

$$T^a x^b = x^{b-a},\ T^a x^a = Tx = x^{10t} - x^{2t} + x^{2t-1}, \text{ for } b > a > 0$$

where $T = x^{10t-1} - x^{2t-1} + Ix^{2t-2}$. Hence, by given equation

$$\begin{aligned} 0 = (T^{10t+1} - T^{2t+1} + T^{2t} - T)x^{8t+3} &= T^{2t-2}(T^{8t+3}x^{8t+3}) - x^{6t+2} + x^{6t+3} - x^{8t+2} \\ &= T^{2t-2}(x^{10t} - x^{2t} + x^{2t-1}) - x^{6t+2} + x^{6t+3} - x^{8t+2} \\ &= x^{8t+2} - x^2 + x - x^{6t+2} + x^{6t+3} - x^{8t+2} = x^{6t+1}(x^2 - x) - (x^2 - x), \end{aligned}$$

that is, by letting $P = x^2 - x$,

$$x^{6t+1}P = P \qquad (1)$$

Let $Q = x^{2t} - x$, then by given equation and by (1)

$$-QP = x^{10t+1}P - x^{2t+1}P = x^{4t}P - x^{2t+1}P = x^{2t}QP.$$

Since Q and P commute, it follows by (1) that

$$PQ = x^{6t+1}PQ = x^{6t+1}QP = x^{4t+1}(-QP) = x^{2t+1}QP = -xQP.$$

Therefore

$$(QP)^2 = (x^{2t}QP - xQP)P = (-QP + PQ)P = 0$$

and, by the first remark, $QP = 0$, that is $x^{2t}P = xP$. Hence by (1)

$$P = x^{6t+1}P = x^{4t+2}P = x^{2t+3}P = x^4P. \qquad (2)$$

Finally, by (1) and (2)

$$P^2 = x^2P - xP = x^{2+(6t-1)\cdot(6t+1)}P - x^{1+9t^2\cdot 4}P = 0$$

which implies that $P = 0$, that is $x^2 = x$.

b) Note that $2x = (2x)^2 = (x + x)^2 = 4x^2 = 4x$ and therefore $2x = 0$, which means that R has characteristic 2. Moreover, by Jacobson's Commutativity Theorem, $x^2 = x$ implies that R is commutative.

Problem 6(Cat A) Marking Scheme

№	Steps	Points
1.	For any r natural if $x^r = 0$ implies $x = 0$……………………………………………………	1pts
2.	Getting (1) identity……………………………………………………………………....	3pts
3.	Showing P and Q commute ……………………………………..……………………..	3pts
4.	Showing $P^2 = 0$ ……………………………………………………………..……………...	1pts
5.	For part b) ………………………………………..……..………………………	2pts

CATEGORY B

Problem 1

For the *10th* anniversary of ***IUHD***, scientists of ***IUHD*** invented "*OMOUScoin*" which is a new cryptocurrency and for testing there are two wallets with a, b positive integer amounts of *OMOUScoins* respectively. There is some hacker who found following weaknesses (operations) of the blockchain:

1. It is possible to steal an equal number of *OMOUScoins* from both of wallets.
2. It is possible to double amount of *OMOUScoins* in any wallet.

Analyze when it's possible to empty both wallets after a finite sequence of operations?

Proposed by Pirmyrat Gurbanov, & Murat Chashemov, IUHD, Ashgabat, Turkmenistan

Solution

We analyze situation any pair m, n

Perform the following algorithm:

1 If $m = n$ remove all the balls from each bag.
2 Assume without loss of generality $m > n$. Remove $n - 1$ balls from each bag.
3 Double the bag with 1 ball in it to get 2 balls.
4 Remove 1 from each bag.
5 Repeat **3** and **4** until both bags have 1 ball.
6 Remove 1 ball from both bags.

It is clear that this algorithm terminates.

Problem 1(Cat A,B) Marking Scheme

№	Steps	Points
1.	Finding algorithm and observing pairing……………………..……………………………..	5pts
2.	Finding invariants and for part B (Cat A) ………………………………………………...	3pts
3.	Empty cases of part A, B (Cat A)…………………………………………………….….	2pts

Problem 2

There are three tangent lines to the parabola at ponits a, b, c and their intersections forming a triangle ABC. Prove the following statements:

a) Circumcircle of ABC passes from the focus of the parabola;

b) Intersection point of the altitudes of the triangle ABC lies on a directrix of the parabola.

c) $S_{abc} = 2S_{ABC}$;

d) $\sqrt[3]{S_{ABC}} + \sqrt[3]{S_{BCA}} = \sqrt[3]{S_{ACB}}$ (here S_{xyz} denotes the the area of triangle formed by points x, y, z).

Proposed by Gurtmyrat Motukov, Eötvös Loránd University, Budapest, Hungary

Solution

a) The projection of the focus F onto the tangent to the parabola lies on the tangent to the parabola perpendicular to the axis. Therefore, the projections A_0, B_0, C_0 focus F on the lines BC, CA, AB lie on the same line. It means that point F lies on the circumcircle of triangle ABC. Indeed, $\angle AFC_0 = \angle AB_0C_0 = \angle A_0B_0C = \angle A_0FC$, so $\angle CFA = \angle A_0FC_0 = 180 - \angle B$.

b) Tangents to the parabola $x^2 = 4y$ at points $(2t_i, t_i^2)$ are given by the equations $y = t_ix - t_i^2$. They intersect at points $(t_i + t_j, t_it_j)$. It's easy to check that the orthocenter of a triangle with vertices at three such points is the point $(t_1 + t_2 + t_3 + t_1t_2t_3, -1)$.

c) We can assume that the parabola is given by the equation $x^2 = 4y$. In such case, points $a, b,$ c have coordinates $(2t_i, t_i^2), i = 1, 2, 3$. It is easy to check that

$$S_{abc} = \frac{1}{2}\begin{vmatrix} 2t_1 & t_1^2 & 1 \\ 2t_2 & t_2^2 & 1 \\ 2t_3 & t_3^2 & 1 \end{vmatrix}, \qquad S_{\mathrm{ABC}} = \frac{1}{2}\begin{vmatrix} t_2 + t_3 & t_2t_3 & 1 \\ t_3 + t_1 & t_3t_1 & 1 \\ t_1 + t_2 & t_1t_2 & 1 \end{vmatrix}$$

d) There is an affine transformation that takes the axis of a parabola and the line AC into a pair of perpendicular lines. Therefore, we can assume that the points a, b, c have coordinates $(2t_1, t_1^2), (0,0), (2t^3, t_3^2)$, with $t_1 < 0$ and $t_3 > 0$.

In this case

$$S_{abC} = -\frac{1}{2}t_1^3, S_{bgA} = \frac{1}{2t_3^3}, \qquad S_{abB} = \frac{1}{2}\begin{vmatrix} 2t_3 & t_3^2 & 1 \\ 2t_1 & t_1^2 & 1 \\ t_1 + t_3 & t_1t_3 & 1 \end{vmatrix} = \frac{(t_3 - t_1)^3}{2}.$$

Problem 2(Cat B) Marking Scheme

№	Steps	Points
1.	For part (a) …………………………………………………….………………………	2pts
2.	For part (b)…………………………………………………………………………..	2pts
3.	For part (c)………………………..…………………………………………….	3pts
4.	For part (d)…………………………………………………………….………..	3pts

Problem3 Find all matrices $A \in M_3(R)$ such that:

$$A^{2025} + A = \begin{pmatrix} 2 & 2 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{pmatrix}$$

Proposed by Rasul Safarov, Novosibirsk State University, Russian Federation

Solution

$$A^{2026} + A^2 = A\begin{pmatrix} 2 & 2 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 2 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{pmatrix}A$$

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}\begin{pmatrix} 2 & 2 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 2 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \Rightarrow \begin{cases} a = e = i \\ d = h = g = 0 \Rightarrow A = \begin{pmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{pmatrix} .So \\ b = f \end{cases}$$

$$A = \begin{pmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{pmatrix} = aI_3 + B,\ where\ B = \begin{pmatrix} 0 & b & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix}, B^2 = \begin{pmatrix} 0 & 0 & b^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, B^3 = O_3$$

$$\Rightarrow\ A^n = (aI_3 + B)^n = \sum_{k=0}^{n} \binom{n}{k} (aI_3)^{n-k} B^k = \binom{n}{0} a^n I_3^n + \binom{n}{1} a^{n-1} I_3^{n-1} B + \binom{n}{2} a^{n-2} I_3^{n-2} B^2$$

$$A^n = a^{n-2}\begin{pmatrix} a^2 & nab & nac + \frac{n(n-1)}{2}b^2 \\ 0 & a^2 & nab \\ 0 & 0 & a^2 \end{pmatrix}$$

$$A^{2025} + A = a^{2023}\begin{pmatrix} a^2 & 2025ab & 2025ac + 2025 \cdot 1012b^2 \\ 0 & a^2 & 2025ab \\ 0 & 0 & a^2 \end{pmatrix} + \begin{pmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{pmatrix} = \begin{pmatrix} 2 & 2 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{pmatrix}$$

$$\begin{cases} a^{2025} + a = 2 \\ 2025a^{2024}b + b = 2 \\ 2025a^{2024}c + 2025 \times 1012a^{2023}b^2 + c = 0 \end{cases} \qquad (1)$$

We have: $a^{2025} + a - 2 = 0$. *Let* $f(a) = a^{2025} + a - 2, f'(a) = 2025a^{2024}$*+1>0, then f(a) increasing. So unique solution a=1. Then* $2025b + b = 2 \Rightarrow b = \frac{1}{1013}$

$$2025c + \frac{2025 \cdot 1012}{1013^2} + c = 0 \Rightarrow c = -\frac{2025 \cdot 506}{1013^3}$$

$$A = \begin{pmatrix} 1 & \frac{1}{1013} & -\frac{2025 \cdot 506}{1013^3} \\ 0 & 1 & \frac{1}{1013} \\ 0 & 0 & 1 \end{pmatrix}$$

Problem 3(Cat B) Marking Scheme

№	Steps	Points
1.	Idea of multiplying by A.........................	1pts
2.	Upper diagonal form for A……………	2pts
2.	General form and and system of equations ….………………......	5pts
3.	Last form of A …………………………………………	1pts

Problem 4
Compute the limit

$$\lim_{n\to\infty} \frac{\sum_{i,j,k=1}^{n} \frac{1}{i+j+k}}{n^2 \ln(n)}.$$

Proposed by Karen Keryan, Yerevan State University, Armenia

Solution
By Cesaro-Schtolz theorem

$$\lim_{n\to\infty} \frac{\sum_{i,j,k=1}^{n} \frac{1}{i+j+k}}{n^2 lnn} = \lim_{n\to\infty} \frac{\sum_{i,j,k=1}^{n} \frac{1}{i+j+k} - \sum_{i,j,k=1}^{n-1} \frac{1}{i+j+k}}{n^2 lnn - (n-1)^2 \ln(n-1)}$$

$$= \lim_{n\to\infty} \frac{3\sum_{i,j=1}^{n-1} \frac{1}{i+j+n} + 3\sum_{i=1}^{n-1} \frac{1}{i+2n} + \frac{1}{3n}}{(n-1)^2 \ln\left(1+\frac{1}{n-1}\right) - (2n-1)\ln(n)} = \frac{3}{2}\lim_{n\to\infty} \frac{\sum_{i,j=1}^{n-1} \frac{1}{i+j+n} + \sum_{i=1}^{n-1} \frac{1}{i+2n}}{nln}$$

Applying Cesaro-Schtolz theorem twice again we get

$$\lim_{n\to\infty} \frac{\sum_{i,j,k}^{n} \frac{1}{i+j+k}}{n^2 \ln(n)} = \frac{3}{2}\lim_{n\to\infty} \frac{\sum_{i,j=1}^{n} \frac{1}{i+j+n} + \sum_{i=1}^{n} \frac{1}{i+2n} - \sum_{i,j=1}^{n-1} \frac{1}{i+j+n} - \sum_{i=1}^{n-1} \frac{1}{i+2n}}{(n+1)\ln(n+1) - nlnn}$$

$$= \frac{3}{2}\lim_{n\to\infty} \frac{\left(2\sum_{i=1}^{n-1} \frac{1}{i+2n} + \frac{1}{3n}\right) + \frac{1}{3n}}{\ln(n+1)} = \frac{3}{2}\lim_{n\to\infty} \frac{2\sum_{i=1}^{n} \frac{1}{i+2n} + 2\sum_{i=1}^{n-1} \frac{1}{i+2n}}{\ln\left(1+\frac{1}{n}\right)} = 3\lim_{n\to\infty} \frac{\frac{1}{3n}}{\frac{1}{n}}$$

$$= 1.$$

Problem 4 (Cat B) Marking Scheme

№	Steps	Points
1.	First use of Stolz-Cesaro theorem……….......…………………………………………..	4pts
2.	Second Use of Stolz-Cesaro theorem ………..	6pts
3.	Here can be some tryings to pass triple integral………………………….	? pts

Note: Here can be some trying's to pass triple integral

№	Steps	Points
1.	Writing Riemann sum …………………………………………………………………	1pts
2.	Getting concrete integral……………………………………………………………….	3pts
3.	Calculating integral final correct answer…………………….	6pts

Problem5

Let $A, B \in M_{2023}(C)$ such that $A^2 = O_{2023}$ and if $ABA = O_{2023}$ then find

$$\det(AB - BA + A + I_{2023})$$

Proposed by Agaserdar Yollyyev, Turkmen State University, Ashgabat, Turkmenistan.

Solution

(a) $(A + AB - BA)^3 = (A + AB - BA)^2(A + AB - BA) = (AB - BA)^2(A + AB - BA) = -AB^2A(A + AB - BA) = O_{2023}$, so $A + AB - BA$ is nilpotent matrix, its all eigenvalues are zero. Then all eigenvalues of $A + AB - BA + I$ matrix are 1, yields:

$$\det(AB - BA + A + I) = 1 \cdot 1 \cdot \ldots \cdot 1 = 1.$$

Problem 5 (Cat B) Marking Scheme

№	Steps	Points
1.	showing $A + AB - BA$ is nilpotent…..….	5pts
2.	Claiming zero eigenvalues and finding det ………………………………..……….…......	5pts

Problem 6

Compute the following integral:

$$\int_0^\infty \frac{sin^4(x)}{x^4}\,dx$$

Proposed by Muhammetbahram Orazov, Heinrich Heine University of Düsseldorf, Germany

Solution

Extended Lobachevsky integral formula:

Let $f(x)$ satisfies in$f(x+\pi)=f(x)$, and $f(x-\pi)=f(x)$, $0\le x\le\infty$. If the following integral

$$\int_0^\infty \frac{\sin^4 x}{x^4} f(x)\,dx$$

defined in the sense of the improper Riemann integral, then we have the following

equality

$$\int_0^\infty \frac{\sin^4 x}{x^4} f(x)\,dx = \int_0^{\frac{\pi}{2}} f(t)dt - \frac{2}{3}\int_0^{\frac{\pi}{2}} \sin^2 t\, f(t)dt.$$

If we take $f(x)=1$ we have

$$\int_0^\infty \frac{\sin^4 x}{x^4} f(x)\,dx = \frac{\pi}{3}.$$

Problem 6(Cat B) Marking Scheme

№	Steps	Points
1.	Lobachevsy theorem ......……………………………….…………	9pts
2.	$\int_0^\infty \frac{\sin^4 x}{x^4} f(x)\,dx = \frac{\pi}{3}$.…...………...	1pts

Note: Here can be some solution using **contour and complex calculus or** just **integrating by parts** also works. In such case we give 10 or 9 or 0 points.

Team Competition

Problem 1

Alice and Bob playing following game: there are 2024 cards and 169 of them is green, 676 of them is red, 441 of them is blue and remained part is purple. At each step Alice chooses a random three cards and if they are with different colors, Bob must change the colors of this cards and the quantity of cards isn't change. If after some operations all the cards have same color Alice wins. Can Alice win in this game?

Proposed by Pirmyrat Gurbanov, & Murat Chashemov, IUHD, Ashgabat, Turkmenistan

Solution

Note that at each step, the parity of the number of cards of each color changes, because either exactly one card of that color is removed, or exactly three cards of that color are added. Therefore, since at the beginning there are two colors with an odd number of cards, and two colors with an even number of cards, at each step in the process there will be two colors with an odd number of cards, and two colors with an even number of cards. The desired final situation has all four numbers of cards of each colors even (three 0, one 2024), and cannot thus be obtained

Problem 1(Team) Marking Scheme (Solution1)

№	Steps	Points
1.	Idea finding invariant…………………………………………………………….……………..	8pts
2.	Full explained starting and final point ……………………………………………….………………...	2pts

Problem 2

Let $A, B \in M_{2023}(C)$ such that $A^2 = O_{2023}$. Find all possible values of :

$$\det(AB - BA).$$

Proposed by Agaserdar Yollyyev, Turkmen State University, Ashgabat, Turkmenistan.

Solution

By Jordan matrix of A, $A^2 = O_n$ yields $rank(A) \leq \left\lfloor \frac{2023}{2} \right\rfloor = 1011$, so:

$$rank(AB - BA) \leq rank(AB) + rank(BA) \leq rank(A) + rank(A) \leq 2022$$

but our matrix has order 2023, so $\det(AB - BA) = 0$.

Problem 2(Team) Marking Scheme

№	Steps	Points
1.	$A^2 = O_n$ and use of Jordan form……………………………………………….……………...	3pts
2.	$rank(A) \leq 1011$ …..…………………………………….………………....	5pts
3.	Showing $rank(AB - BA)$………………………………………………………..	2pts

Problem 3

Let $P_n = n!\,(n-1)!\,(n-2)!\dots 2!\,1!$. Let p be a prime number. Let $d_n = d(n)$ be the number of p's in the prime factorization of P_n. Show that

$$d(n) < \frac{n(n+1)}{2(p-1)}.$$

Proposed by Yagub Aliyev, ADA University, Baku, Azerbaijan

Solution

By Legendre's formula [1], which is obvious, the number of ps in prime factorization of $n!$, for prime p, is

$$\nu_p(n!) = \left\lfloor\frac{n}{p}\right\rfloor + \left\lfloor\frac{n}{p^2}\right\rfloor + \left\lfloor\frac{n}{p^3}\right\rfloor + \cdots + \left\lfloor\frac{n}{p^k}\right\rfloor + \cdots < \frac{n}{p} + \frac{n}{p^2} + \frac{n}{p^3} + \cdots + \frac{n}{p^k} + \cdots = \frac{n}{p-1},$$

where $\nu_p(x) = \mathrm{val}_p(x)$ is the p-adic valuation function, counting the number of ps in x. Then

$$d(n) = \nu_p(n!\,(n-1)!\,(n-2)!\dots 2!\,1!) < \frac{1}{p-1} + \frac{2}{p-1} + \frac{3}{p-1} + \cdots + \frac{n}{p-1}$$
$$= \frac{n(n+1)}{2(p-1)}.$$

Problem 3(Team) Marking Scheme

№	Steps	Points
1.	Using Legendre formula……………………………………………………………...	8pts
2.	Lower bound for $d(n)$ ………………………………....	2pts

Problem 4

Two groups started fighting. The process is described by the linear system

$$\begin{cases} \dot{x} = -y, \\ \dot{y} = -x, \end{cases}$$

with $x(t)$ and $y(t)$ denoting, at any moment t, the number of undefeated fighters in the groups. So, the losses in each group are proportional to the number of its opponents.

The first group initially has X fighters. Due to their advantageous position, they can split the opponent Y fighters into $Y_1 + Y_2$, at any proportion. Thus, X fighters start against Y_1 opponents, and then, after their victory, the rest of them continue against Y_2.

What is the maximum of Y that can be defeated by X fighters?

Proposed by Irina Astashova, Lomonosov State University, Moscov, Russian Federation.

Solution

We begin by solving

$$\begin{cases} \dot{x} = -y, \\ \dot{y} = -x. \end{cases}$$

Let's get rid of t.

$$\frac{dy}{dt}\frac{dt}{dx} = \frac{x}{y}, \Rightarrow \; y^2 - x^2 = C.$$

So, during combat x and y are decreasing, but $y^2 - x^2 = C$ always holds. If $C > 0$ there's solution $x = 0, y = \sqrt{C}$, that means y is winning side. If $C < 0$, then x is winning side. If $C = 0$ then $x = 0, y = 0$ at the same time, and that means a tie. For a tie we need $y^2 - x^2 = 0$, or $y = x$. Otherwise numerical advantage decides winner. Initial data for the first fight: $x(0) = X, y(0) = Y_1$. We use $y^2 - x^2 = C$ to find casualties of x. Constant C is defined by initial data.

$$y^2 - x^2 = C,$$

$$Y_1^2 - X^2 = C,$$

$$y^2 - x^2 = Y_1^2 - X^2$$

If $y = 0$ (so when x wins) then:

$$x = \sqrt{X^2 - Y_1^2}$$

Second fight $-x$ have $\sqrt{X^2 - Y_1^2}$ remaining fighters, y have Y^2 fighters. To have a tie we need $x = y$, so $Y_2 = \sqrt{X^2 - Y_1^2}$. It's maximum number that rest of x can defeat.

Let's find maximum possible Y.

$$Y = Y_1 + Y_2 = Y_1 + \sqrt{X^2 - Y_1^2}.$$

Differentiate by Y_1.

$$\frac{d\left(Y_1 + \sqrt{X^2 - Y_1^2}\right)}{dY_1} = 1 - \frac{2Y_1}{2\sqrt{X^2 - Y_1^2}}.$$

Derivative is zero when

$\frac{Y_1}{X^2 - Y_1^2} = 1, \Leftrightarrow Y_1^2 = X^2 - Y_1^2$. That means $Y_1 = \frac{X}{\sqrt{2}}$. Substituting Y_1 into the formula for Y_2 we obtain $Y_2 = \frac{X}{\sqrt{2}}$. So here's the answer: $Y = X\sqrt{2}$ is a tie, if $Y < X\sqrt{2}$ then X can win, enemy forces better be split in half.

Problem 4(Team) Marking Scheme

№	Steps	Points
1.	$y^2 - x^2 = C$ ………………………………………………………………………………...	1pts
2.	$y^2 - x^2 = Y_1^2 - X^2$………………………………………………………………….....	3pts
3.	Consider $y = 0$ and $Y = Y_1 + Y_2 = Y_1 + \sqrt{X^2 - Y_1^2}$ ………………………………………...	3pts
4.	Final answer and finding maximum.…………………	3pts

Problem 5 A matrix $P \in M_n(C)$ is called a projection if $P^2 = P$. Find all linear mappings $\varphi: M_n(C) \to R$ such that $\varphi(P) = rank\ (P)$ for every projection $P \in M_n(C)$.

Proposed by Asen Bozhilov and Stoyan Apostolov, Sofia University St. Kliment Ohridski, Sofia, Bulgaria.

Solution

Let φ be such mapping. Let E_{ij} denote a matrix with only one nonzero entry - the

element positioned on the i-th row and j-th column is 1. It is directly verified that the matrices of the form $E_{ij} + E_{ii}$ with $i \neq j$ and E_{ii} are projections. Let W be the set of all such matrices.

Clearly the linear span of W, $\ell(W)$, contains E_{ij} and E_{ii}, hence $\ell(W) = M_n(C)$. Observe that $\varphi(A) = rank(A) = 1 = tr(A)$ for every $A \in W$. Thus $tr : M_n(C) \to C$ is a linear mapping which coincides with φ on W. When two linear mappings coincide on some set, then they coincide on the linear span of the set. This shows that $\varphi \equiv tr$ on $\ell(W) = M_n(C)$. It remains to observe that $tr(P) = rank(P)$ for every projection P. Indeed, the minimal polynomial of a projection divides $x(x-1)$, hence it has simple roots. Thus every projection is diagonalizable. Hence the rank of a projection is equal to the number of nonzero charachteristic roots it has. On the other hand, the nonzero roots of a projection are equal to 1, hence their number is actually their sum, which equals the trace of the projection.

Problem 5(Team) Marking Scheme

№	Steps	Points
1.	Considering $E_{ij} + E_{ii}$, E_{ii} and verifying that they are projectors ………………………	2pts
2.	$\ell(W) = M_n(C)$. ………………….……………....	1pts
3.	Observing that $\varphi(A) = rank(A) = 1 = tr(A)$………………..	2pts
4.	Showing that $\varphi \equiv tr$ on $\ell(W)$…………………………………………………………..	1pts
5.	$tr(P) = rank(P)$ for every projection P…………………………………….	3pts
6.	Observing characteristic roots and minimal polynomial $x(x-1)$…………………………..	2pts

Problem 6 Compute the sum

$$\sum_{n=0}^{\infty} \frac{((2n)!)^2}{(4n+1)!}$$

Proposed by Karen Keryan, Yerevan State University, Armenia

Solution

By relationship between the beta and the gamma functions

$$S := \sum_{n=0}^{\infty} \frac{\left((2n)!\right)^2}{(4n\ +\ 1)!} = \sum_{n=0}^{\infty} \frac{\Gamma^2(2n\ +\ 1)}{\Gamma(4n\ +\ 2)} = \sum_{n=0}^{\infty} B(2n\ +\ 1, 2n\ +\ 1) = \sum_{n=0}^{\infty} \int_0^1 \left(x(1\ - x)\right)^{2n} dx.$$

Since the series $\sum_{n=0}^{\infty} \int_0^1 \left(x(1\ - x)\right)^{2n} dx$ is uniformly convergent on $[0, 1]$ (by Weierstrass M-test) we get

$$S = \int_0^1 \sum_{n=0}^{\infty} \left(x(1\ -\ x)\right)^{2n} dx = \int_0^1 \frac{1}{1 - \left(x(1\ -\ x)\right)^2} dx$$

$$= \frac{1}{2} \int_0^1 \left(\frac{1}{1 - x(1-x)} + \frac{1}{1 + x(1-x)} \right) dx$$

$$= \frac{1}{2} \int_0^1 \left(\frac{1}{\left(x - \frac{1}{2}\right)^2 + \frac{3}{4}} + \frac{1}{\left(x - \frac{1}{2}\right)^2 - \frac{5}{4}} \right) dx$$

$$= \left(\frac{1}{\sqrt{3}} arctg \frac{2x-1}{\sqrt{3}} - \frac{1}{2\sqrt{5}} ln \left| \frac{2x - 1 - \sqrt{5}}{2x - 1 + \sqrt{5}} \right| \right) \Big|_0^1$$

$$= \frac{\pi}{3\sqrt{3}} - \frac{2\left(\ln(\sqrt{5} - 1) - \ln(2)\right)}{\sqrt{5}}.$$

Problem 6(Team) Marking Scheme

№	Steps	Points
1.	Using Gamma function…………………………….……………	3pts
2.	Using Beta function ….……………………….…..	1pts
3.	Changing Beta to integral………………………………	1pts
4.	Showing uniform convergence of $\sum_{n=0}^{\infty} \int_0^1 \left(x(1\ - x)\right)^{2n} dx$ …………………………………..	3pts
5.	Calculation of integral and answer	2pts

Problem 7

Let 1234 and 1235 be two tetrahedra with a common base of 123, and let l_{ij} be the length of the edge connecting vertices with numbers i and j. Next, let S_{123} be the area of the base of the tetrahedra, and V_{1234} and V_{1235} be the volumes of the given tetrahedra. Prove that the distance d_{45} between vertices 4 and 5 can be calculated by the formula:

$$d_{45}^2 = l_{34}^2 + l_{35}^2 + \frac{D + 36\varepsilon V_{1234}V_{1235}}{2S_{123}^2},$$

where

$$D = \frac{1}{8}\begin{vmatrix} 2l_{23}^2 & -l_{12}^2 + l_{23}^2 + l_{13}^2 & -l_{25}^2 + l_{35}^2 + l_{23}^2 \\ -l_{12}^2 + l_{23}^2 + l_{13}^2 & 2l_{13}^2 & l_{35}^2 + l_{13}^2 - l_{15}^2 \\ -l_{24}^2 + l_{23}^2 + l_{34}^2 & l_{13}^2 + l_{34}^2 - l_{14}^2 & 0 \end{vmatrix}$$

and $\varepsilon = \pm 1$ depends on whether the tops 4 and 5 lie on different or one side of the base 123.

Proposed by Vladimir Krasnov, RUDN university, Russian Federation

Solution

Let us denote by A and B the projections of the vertices 4 and 5 onto the plane of the triangle 123, by h_4 and h_5 the vertices of the tetrahedrons under consideration, and by -$\overrightarrow{l_{ij}}$ — vectors directed from vertex i to vertex j. Then

$$d_{45}^2 = |AB|^2 + |h_4 + \varepsilon h_5|^2.$$

In its turn,

$$|AB| = |\overrightarrow{l_{34}}| \sin\alpha\,,$$

where $\alpha = \angle(5,4,A) = \angle(B,5,4)$ be an angle between $\overrightarrow{l_{54}}$ and $\frac{[\overrightarrow{l_{32}}\overrightarrow{l_{32}}]}{2S_{123}}$. Then

$$AB = \frac{[\overrightarrow{l_{32}}[\overrightarrow{l_{32}}\overrightarrow{l_{32}}]]^2}{4S_{123}^2}, \quad h_4 = \frac{3V_{1234}}{S_{123}}, \quad h_5 = \frac{3V_{1235}}{S_{123}}$$

Besides, $\overrightarrow{l_{54}} = \overrightarrow{l_{53}} + \overrightarrow{l_{34}}$, therefore, according to the double vector product formula, we get

$$|AB|^2 = l_{34}^2 + l_{35}^2 - \frac{(\overrightarrow{l_{35}}\overrightarrow{l_{32}}\overrightarrow{l_{31}})^2}{4S_{123}^2} - \frac{(\overrightarrow{l_{34}}\overrightarrow{l_{32}}\overrightarrow{l_{31}})^2}{4S_{123}^2} + \frac{[\overrightarrow{l_{53}}[\overrightarrow{l_{32}}\overrightarrow{l_{31}}]][\overrightarrow{l_{34}}[\overrightarrow{l_{32}}\overrightarrow{l_{31}}]]}{2S_{123}^2} = l_{34}^2 + l_{35}^2 - \frac{9V_{1234}^2}{S_{123}^2} + \frac{9V_{1235}^2}{S_{123}^2} +$$
$$\frac{1}{2S_{123}^2}\left(\left(\overrightarrow{l_{31}}(\overrightarrow{l_{53}}\overrightarrow{l_{32}})\right) - \left(\overrightarrow{l_{32}}(\overrightarrow{l_{53}}\overrightarrow{l_{31}})\right)\right)\left(\left(\overrightarrow{l_{31}}(\overrightarrow{l_{34}}\overrightarrow{l_{32}})\right) - \left(\overrightarrow{l_{32}}(\overrightarrow{l_{34}}\overrightarrow{l_{31}})\right)\right)$$

Finally, expressing (using the cosine theorem) the scalar products in terms of the squared lengths of the sides, after some simple calculations we arrive at the required formula.

Problem 7(Team) Marking Scheme

№	Steps	Points
1.	$d_{45}^2 = \lvert AB\rvert^2 + \lvert h_4 + \varepsilon h_5\rvert^2$ …………………………………………………………	2pts
2.	For AB, h_4, h_5……………………………………………………………………………………	3pts
3.	Finally answering……………………………………………………………………	4pts

Problem 8

Let $f \in C([0,1],R)$ be a piecewice polynomial, i.e., there exist $n \in N, x_0, x_1, x_2, \dots, x_n \in R$ such that for all $j \in \{1,2,3,\dots,n\}$ it holds that $0 = x_0 < x_1 < \cdots < x_n = 1$, $f|[x_i, x_{i+1}]$ is a polynomial, and f is continuous. Then prove that

$$\left\{q \in (0,1): \frac{(1-q)^2}{6q}\int_0^q f(x)dx = \int_q^1 \left(\frac{q+2}{3} - x\right) f(x)dx \ and \ \int_0^1 f(x)dx \neq \frac{1}{q}\int_0^q f(x)dx\right\}$$

is a finite set

Proposed by Shokhrukh Ibragimov University of Münster, Münster, Germany

Solution

Throughout this proof let $n \in N,\ d_1, d_2, \dots, d_n \in N_0,\ x_0, x_1, x_2, \dots x_n \in R$ satisfy for all $j \in \{1, 2, \dots, n\}$ that $0 = x_0 < x_1 < \dots < x_n = 1,\ f|\left[x_{j-1}, x_j\right]$ is a polynomial, and $d_j = \deg\left(f|\left[x_{j-1}, x_j\right]\right)$ and let $j \in \{1, 2, \dots, n\}$, $q \in \left[x_{j-1}, x_j\right]$ satisfy

$$\frac{(1-q)^2}{6q}\int_0^q f(x)dx = \int_q^1 \left(\frac{q+2}{3} - x\right) f(x)dx \quad and \quad \int_0^1 f(x)dx \neq \frac{1}{q}\int_0^q f(x)dx \qquad (2)$$

First assume that $d_j = 0$. This implies that there exists $c \in R$ such that for all $x \in \left[x_{j-1}, x_j\right]$ it holds that $f(x) = c$. Therefore, we obtain that

$$(1-q)^2\left[\int_0^q f(x)dx\right] = (1-q)^2\left[\int_0^{x_{j-1}} f(x)dx + \int_{x_{j-1}}^q c\,dx\right] \quad \text{and}$$

$$2q\int_q^1 (q+2-3x)f(x)dx = 2q\left[\int_q^{x_j} (q+2-3x)c\,dx + \int_{x_j}^1 (q+2-3x)f(x)dx\right].$$

Combining this with (2) assures that

$$(1-q)^2\left[\int_0^{x_{j-1}} f(x)dx + \int_{x_{j-1}}^q c\,dx\right] = 2q\left[\int_q^{x_j} (q+2-3x)c\,dx + \int_{x_j}^1 (q+2-3x)f(x)dx\right] \qquad (3).$$

Hence, we obtain that

$$q^2\left[\int_0^{x_{j-1}} f(x)\,dx - cx_{j-1} + 2c - 2cx_j - 2\int_{x_j}^{1} f(x)dx\right] +$$
$$+q\left[c + 2cx_{j-1} - 4cx_j + 3c\left[x_j\right]^2 - 2\int_0^{x_{j-1}} f(x)\,dx - 4\int_{x_j}^{1} f(x)\,dx + 6\int_{x_j}^{1} x\cdot f(x)\,dx\right] +$$
$$+\left[\int_0^{x_{j-1}} f(x)\,dx - cx_{j-1}\right] = 0 \qquad (4).$$

Assume that

$$\left[\int_0^{x_{j-1}} f(x)\,dx - cx_{j-1} + 2c - 2cx_j - 2\int_{x_j}^{1} f(x)dx\right]^2 + \left[\int_0^{x_{j-1}} f(x)\,dx - cx_{j-1}\right]^2 = 0 \qquad (5).$$

This implies that

$$\int_0^{x_{j-1}} f(x)\,dx = cx_{j-1} \quad and \quad \int_{x_j}^{1} f(x)\,dx = c - cx_j \qquad (6).$$

Therefore, we obtain that

$$\int_0^1 f(x)\,dx - \frac{1}{q}\int_0^q f(x)\,dx =$$
$$= \int_0^{x_{j-1}} f(x)\,dx + \int_{x_{j-1}}^{x_j} c\,dx + \int_{x_j}^{1} f(x)\,dx - \frac{1}{q}\left[\int_0^{x_{j-1}} f(x)\,dx + \int_{x_{j-1}}^{q} c\,dx\right] =$$
$$= cx_{j-1} + c\left(x_j - x_{j-1}\right) + c - cx_j - \frac{1}{q}\left[cx_{j-1} + c\left(q - x_{j-1}\right)\right] = 0 \qquad (7).$$

This contradicts to the assumption that $\int_0^1 f(x)\,dx \neq \frac{1}{q}\int_0^q f(x)\,dx$. Hence, we obtain that

$$\left[\int_0^{x_{j-1}} f(x)\,dx - cx_{j-1} + 2c - 2cx_j - 2\int_{x_j}^{1} f(x)dx\right]^2 + \left[\int_0^{x_{j-1}} f(x)\,dx - cx_{j-1}\right]^2 \neq 0 \qquad (8).$$

This ensures that the equation in (4) has at most $2 = d_j + 2$ solution on $q \in \left[x_{j-1}, x_j\right]$. Next assume that $d_j > 0$. This assures that there exist a constant $a \in R/\{0\}$ and a polynomial Q with $\deg(Q) \leq d_j - 1$ such that for all $x \in \left[x_{j-1}, x_j\right]$ it holds that $f(x) = ax^{d_j} + Q(x)$. Hence, we obtain that

$$(1-q)^2\left[\int_0^q f(x)\,dx\right] = (1-q)^2\left[\int_0^{x_{j-1}} f(x)\,dx + \int_{x_{j-1}}^{q}\left(ax^{d_j} + Q(x)\right)dx\right]$$ and

$$2q\int_q^1 (q+2-3x)\, f(x)\,dx = 2q\left[\int_q^{x_j} (q+2-3x)\left(ax^{d_j}+Q(x)\right)dx + \int_{x_j}^1 (q+2-3x)\, f(x)\,dx\right].$$

Combining this with (2) shows that

$$(1-q)^2\left[\int_0^{x_{j-1}} f(x)\,dx + \int_{x_{j-1}}^{q}\left(ax^{d_j}+Q(x)\right)dx\right] =$$

$$= 2q\left[\int_q^{x_j} (q+2-3x)\left(ax^{d_j}+Q(x)\right)dx + \int_{x_j}^1 (q+2-3x)\, f(x)\,dx\right] \qquad (9).$$

This ensures that there exists a polynomial $\tilde{Q}$ with $\deg\left(\tilde{Q}\right) \le d_j + 2$ such that

$$q^{d_j+3}\left[\frac{3ad_j}{\left(d_j+1\right)\left(d_j+2\right)}\right] + \tilde{Q}(q) = 0 \qquad (10).$$

Observe that the equation in (10) has at most $d_j + 3$ solutions on $q \in \left[x_{j-1}, x_j\right]$. Therefore, we obtain that for all $j \in \{1, 2, ..., n\}$ it holds that the equation in (2) has at most $d_j + 3$ solutions on $q \in \left[x_{j-1}, x_j\right]$. This ensures that the equation in (2) has at most $d_1 + d_2 + ... + d_n + 3n$ solutions on $q \in (0,1)$. This completes the solution of the problem.

Problem 9(Team) Marking Scheme

№	Steps	Points
1.	Assuming $d_j = 0$ …………………………………………………………...………………	1pts
2.	$2q\int_q^1 (q+2-3x)\, f(x)\,dx = 2q\left[\int_q^{x_j} (q+2-3x)c\,dx + \int_{x_j}^1 (q+2-3x)\, f(x)\,dx\right]$ ……………………………….	1pts
3.	Getting (3) …………………………………………………...…….	1pts
4.	Getting (4) …………………………………………………………………………….	1pts
5.	Establishing (5)+(6)……………………..	2pts
6.	Getting (7) …………..………………………………………………………..….	1pts
7.	Obtaining $2q\int_q^1 (q+2-3x)\, f(x)\,dx = 2q\left[\int_q^{x_j} (q+2-3x)\left(ax^{d_j}+Q(x)\right)dx + \int_{x_j}^1 (q+2-3x)\, f(x)\,dx\right].$ ………….	2pts
8.	Getting (10)…………………………………………………………………….	2pts

Problem 9

In the space of continuously differentiable R functions, the operator is given $\varphi[f(x)](x) = f'(x) + f(x)$. Find $\phi^{2025}\left[\frac{\sin x + x\cos x}{2^{1012}}\right]\left(\frac{\pi}{4}\right)$.

Proposed by Anvar Ibragimov_Uzbek-Finnish Pedagogical Institute, Samarkand, Uzbekistan

Solution

This operator is linear in a linear four-dimensional space with a basis

$e = (\sin x \quad \cos x \quad x\sin x \quad x\cos x)$.

$\varphi[\sin x](x) = \cos x + \sin x = 1\cdot \sin x + 1\cdot \cos x + 0\cdot x\sin x + 0\cdot x\cos x = e(1 \ \ 1 \ \ 0 \ \ 0)^T$;

$\varphi[\cos x](x) = -\sin x + \cos x = (-1)\cdot \sin x + 1\cdot \cos x + 0\cdot x\sin x + 0\cdot x\cos x = e(-1 \ \ 1 \ \ 0 \ \ 0)^T$;

$\varphi[x\sin x](x) = \sin x + x\cos x + x\sin x = 1\cdot \sin x + 0\cdot \cos x + 1\cdot x\sin x + 1\cdot x\cos x = e(1 \ \ 0 \ \ 1 \ \ 1)^T$;

$\varphi[x\cos x](x) = \cos x - x\sin x + x\cos x = 0\cdot \sin x + 1\cdot \cos x + (-1)\cdot x\sin x + 1\cdot x\cos x = e(0 \ \ 1 \ \ -1 \ \ 1)^T$.

Therefore, the transformation matrix φ in basis e has the form

$$A = \begin{pmatrix} 1 & -1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix} = \underbrace{\sqrt{2}\cdot \begin{pmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} & 0 & 0 \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} & 0 & 0 \\ 0 & 0 & \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ 0 & 0 & \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{pmatrix}}_{B} + \underbrace{\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}}_{C}$$

Note that $\frac{1}{\sqrt{2}}B$ is a block-diagonal matrix, on the diagonals of which there are rotation matrices by angle $\frac{\pi}{4}$, so

$$B^n = 2^{\frac{n}{2}}\cdot \begin{pmatrix} \cos\frac{\pi n}{4} & -\sin\frac{\pi n}{4} & 0 & 0 \\ \sin\frac{\pi n}{4} & \cos\frac{\pi n}{4} & 0 & 0 \\ 0 & 0 & \cos\frac{\pi n}{4} & -\sin\frac{\pi n}{4} \\ 0 & 0 & \sin\frac{\pi n}{4} & \cos\frac{\pi n}{4} \end{pmatrix}$$

$A^n = (B+C)^n = B^n + nB^{n-1}C$, because $C^2 = 0$. So,

$$A^n = 2^{\frac{n-1}{2}} \cdot \begin{pmatrix} \sqrt{2}\cos\frac{\pi n}{4} & -\sqrt{2}\sin\frac{\pi n}{4} & n\cdot\cos\frac{\pi(n-1)}{4} & -n\cdot\sin\frac{\pi(n-1)}{4} \\ \sqrt{2}\sin\frac{\pi n}{4} & \sqrt{2}\cos\frac{\pi n}{4} & n\cdot\sin\frac{\pi(n-1)}{4} & n\cdot\cos\frac{\pi(n-1)}{4} \\ 0 & 0 & \sqrt{2}\cos\frac{\pi n}{4} & -\sqrt{2}\sin\frac{\pi n}{4} \\ 0 & 0 & \sqrt{2}\sin\frac{\pi n}{4} & \sqrt{2}\cos\frac{\pi n}{4} \end{pmatrix}$$

$$\varphi^{2025}\left[\frac{\sin x + x\cos x}{2^{1012}}\right](x) = e\cdot 2^{-1012}\cdot A^{2025}\begin{pmatrix}1\\0\\0\\1\end{pmatrix} = e\begin{pmatrix} \sqrt{2}\cos\frac{2025\pi}{4} - 2025\sin\frac{2024\pi}{4} \\ \sqrt{2}\sin\frac{2025\pi}{4} + 2025\cos\frac{2024\pi}{4} \\ -\sqrt{2}\sin\frac{2025\pi}{4} \\ \sqrt{2}\cos\frac{2025\pi}{4} \end{pmatrix} = e\begin{pmatrix}1\\2026\\-1\\1\end{pmatrix}$$

Total

$$\varphi^{2025}\left[\frac{\sin x + x\cos x}{2^{1012}}\right]\left(\frac{\pi}{4}\right) = 1\cdot\sin\frac{\pi}{4} + 2026\cdot\cos\frac{\pi}{4} - 1\cdot\frac{\pi}{4}\sin\frac{\pi}{4} + 1\cdot\frac{\pi}{4}\cos\frac{\pi}{4} = \frac{2027\sqrt{2}}{2} = \frac{2027}{\sqrt{2}}.$$

Problem 8(Team) Marking Scheme

№	Steps	Points
1.	Considering basis for operator…………………………………………..…………………	2pts
2.	Using operator with trigonometric functions…………………………………………………	1pts
3.	Writing in form $A = B + C$…………………………………………………………….……..	1pts
4.	Finding B^n………………………….…………………………………………………….….	2pts
5.	$C^2 = 0$ ………………………...	1pts
6.	For A^n in explicit form………………………………………………….	1pts
7.	Finding $\varphi^{2025}\left[\frac{\sin(x)+x cos(x)}{2^{1012}}\right]\left(\frac{\pi}{4}\right)$ …..……………………………………………..	2pts
8.	Final answer………………………………………………………………………………	1

Problem 10

Does there exists a dense set in the plane, such that no three of its points lie on the same line?

Proposed by Asen Bozhilov and Stoyan Apostolov, Sofia University St. Kliment Ohridski, Sofia, Bulgaria.

Solution

Let $\{(x_n, y_n)\}_{n\geq 1}$ with $x_n \in Q$, $y_n \in Q$ for all n, be a sequence of points which is dense in R^2 and $x_n \neq x_m$ when $n \neq m$. Such a sequence could be constructed as follows: let

$\{V_n\}_{n\geq 1}$ be an enumeration of a base for the standard topology in R^2, for example the balls whose centers have rational coordinates, and whose radii are positive rational numbers. Clealy, each V_n contains infinitely many points with rational coordinates (since these points are dense in R^2). Now construct $\{(x_n, y_n)\}_{n\geq 1}$ inductively - if (x_k, y_k) has been constructed for $k \leq n-1$, then choose (x_n, y_n) from V_n in such a way that $x_n \neq x_k$ for $k < n$. This sequence has elements in all of the V_n, hence is dense in R^2. Now let θ be a transcendental number with $\theta \in (0,1)$ (for example $\theta = \frac{1}{e}$). Consider the sequence $\{(x_n, \widetilde{y_n})\}_{n\geq 1}$ where $\widetilde{y_n} = y_n + \theta_n$. We claim that this set satisfies the problem statement. First we show that $\{(x_n, \tilde{y}_n)\}_{n\geq 1}$ is dense in R^2. Indeed, let $(a,b) \in R^2$ and $\varepsilon > 0$. Since $\{(x_n, y_n)\}_{n\geq 1}$ is dense in R^2, there exist infinitely many j such that $\| (x_j, y_j) - (a,b) \| < \frac{\varepsilon}{2}$. Choose such j_0, which is also large enough so that $\theta^{j_0} < \frac{\varepsilon}{2}$. Thus $\| \left(x_{j_0}, \widetilde{y_{J_0}}\right) - (a,b) \| \leq \frac{\varepsilon}{2}$.

Now assume that there exist three distinct points $(x_m, \tilde{y}_m)$, $(x_k, \tilde{y}_k)$, and $(x_l, \tilde{y}_l)$ which lie on

the same line, i.e. there exists $\alpha \in R$ such that $\alpha(x_m, y_m) + (1-\alpha)(x_k, \tilde{y}_k) = (x_l, \tilde{y}_l)$. Comparing the first coordinates we obtain that α is uniquely determined nonzero rational number (since $x_m \neq x_k$ and $x_k \neq x_l$). Comparing the second coordinates yields

$$\alpha y_m + \alpha\theta_m + (1-\alpha)y_k + (1-\alpha)\theta^k = y_l + \theta_l.$$

Thus, if we consider the polynomial $\alpha x_m + (1-\alpha)x_k - x_l + r$, where

$$r = \alpha y_m + (1-\alpha)y_k - y_l,$$

we see that this is a nonzero polynomial with rational coefficients, and θ is one of its roots. This is in contradiction with θ being transcendental.

Problem 10(Team) Marking Scheme

№	Steps	Points
1.	Working idea about construction of example ………………………..……………………	1pts
2.	For working construction with small mistakes……………..	9pts